International Journal of Theoretical Physics, Vol. 31, No. 5, 1992

Regular Measures and Inner Product Spaces

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Received October 17, 1991

We present some results concerning the properties of regular finitely additive measures on the set of all splitting subspaces of a (real or complex) inner product space S and their relation to completeness of S. These results are generalized for abstract quadratic spaces to be orthomodular. Moreover, some open problems are presented.

1. INTRODUCTION

One of the basic problems related to the mathematical foundations of quantum mechanics (Birkhoff and von Neumann, 1936; Mackey, 1963; von Neumann, 1932) is the description of probability measures (called states in physical terminology) on the set of experimentally verifiable propositions regarding a physical system [or psychology of the human brain, computer science, or sociometry; for details, see Grib *et al.* (1989)]. The set of propositions forms an orthomodular, orthocomplemented poset which is called a quantum logic. In the more restrictive setting a quantum logic is assumed to be a complete orthomodular lattice (Varadarajan, 1968).

An important interpretation of a quantum logic is via the set $L(H)$ of all closed subspaces of a real or complex Hilbert space H, with an inner product $(\cdot, \cdot)$, which is an orthomodular, complete lattice with respect to the set-theoretic inclusion and the natural orthocomplementation $\perp: M \mapsto M^\perp = \{x \in H: (x, y) = 0 \text{ for each } y \in M\}$. In this model, a finite, countably additive measure is any mapping $m: L(H) \to [0, \infty)$ such that

$$m(\textstyle\bigvee_{i=1}^{\infty} M_i) = \sum_{i=1}^{\infty} m(M_i)$$

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0020-7748/92/0500-0889$06.50/0

for any sequence of mutually orthogonal subspaces $\{M_i\}$ of $L(H)$. The cornerstone of this description is a famous result of Gleason (1957) which says that countably additive measures on $L(H)$, where H is a separable Hilbert space and dim $H \neq 2$, are in a one-to-one correspondence with positive operators T of the trace class on H via

$$m(M) = \mathrm{tr}(TP^M), \qquad M \in L(H) \tag{1}$$

where P^M denotes the orthoprojector from H onto M. This result has been generalized also for nonseparable Hilbert spaces by Eilers and Horst (1975), Drisch (1979), and Maeda (1980), and for bounded signed measures by Sherstnev (1974).

If the assumption of the completeness of H is omitted, we obtain a more general class of real or complex inner product spaces which can be used as axiomatic models. In this connection two families of closed subspaces play an important role: If S is an inner product space, then by $E(S)$ we denote the set of all splitting subspaces of S, i.e., of all subspaces M of S for which the projection theorem $M + M^\perp = S$ holds. This is an orthocomplemented, orthomodular poset containing $\{O\}$ and S and also any complete subspace and therefore, all finite-dimensional subspaces of S.

By $F(S)$ we denote the set of all orthogonally closed subspaces M of S, i.e., of all subspaces M of S for which $M = M^{\perp\perp}$. It is well known that $F(S)$ is an orthocomplemented, complete lattice, and $E(S) \subseteq F(S)$.

An interesting algebraic characterization of the completeness of S is due to Amemiya and Araki (1966–1967), which says that S is complete iff $F(S)$ is orthomodular, or, equivalently, if $E(S) = F(S)$. Dvurečenskij (1988) proved that S is complete iff $E(S)$ is a σ-logic, i.e., if $E(S)$ possesses the join of any sequence of mutually orthogonal splitting (it suffices one-dimensional) subspaces of S.

The measure-theoretic completeness characterizations were begun by Hamhalter and Pták (1987), showing that a separable S is complete iff $F(S)$ possesses at least one probability measure. This result for $E(S)$, $F(S)$, and other families of subspaces for a general S, as well as for signed measures and frame functions, has been generalized by Dvurečenskij (1989*a*,*b*), Dvurečenskij and Mišík (1988), Dvurečenskij and Pulmannová (1988, 1989), and Dvurečenskij *et al.* (1990). Dvurečenskij (1991, and to appear) proved that S is complete iff $F(S)$ possesses at least one regular finitely additive measure, i.e., a measure which is approximable from below by finite-dimensional subspaces.

On the other hand, any $E(S)$ possesses plenty of regular finitely additive measures even for incomplete S.

Therefore, in this paper, we shall investigate conditions on systems of regular finitely additive measures on $E(S)$ which will guarantee the completeness of S. It will be shown that for completeness criteria we must take plenty of regular finitely additive measures. Moreover, it will be shown that these results can be generalized to more general quadratic spaces, i.e., inner product spaces not necessarily over the field of real or complex numbers. These spaces have been studied, e.g., by Keller (1980, 1990), Gross (1990), Gross and Keller (1977), Piziak (1990), and Kalmbach (1990). This space is said to be orthomodular if $E(S)=F(S)$. Using the measure-theoretic characterizations, we shall investigate the orthomodularity of quadratic spaces.

2. REGULAR MEASURES

Let S be a real or complex inner product space. A mapping $m: E(S) \to R$ such that

$$m\left(\bigoplus_{i\in I} M_i\right)=\sum_{i\in I} m(M_i) \tag{2}$$

whenever $\{M_i : i\in I\}$ is a system of mutually orthogonal subspaces of $E(S)$ for which the join $\bigoplus_{i\in I} M_i$ exists in $E(S)$, is said to be a *charge*, *signed measure*, or *completely additive signed measure* if (2) holds for any finite, countable, or arbitrary index set I [the latter case means that the real net $\{\sum_{i\in D} m(M_i): \mathrm{D}$ is a finite subset of $I\}$ converges in R with the limit $m(\bigoplus_{i\in I} M_i)$]. If m attains only positive values, we say that m is a *finitely additive measure*, *measure*, or *completely additive measure*, respectively, according to the cardinality of I. A charge is said to be *Jordan* if it can be represented as a difference of two positive finitely additive measures.

Let $P(S)$ be the set of all finite-dimensional subspaces of S. A charge m on $E(S)$ is said to be *$P(S)$-regular* if given $M\in E(S)$ and given $\varepsilon>0$ there exists a finite-dimensional subspace N of M such that

$$|m(M\cap N^{\perp})|<\varepsilon \tag{3}$$

[We recall that if $N\subseteq M$, N, $M\in E(S)$, then $M\cap N^{\perp}\in E(S)$.]

All the above notions can be defined in the same way also for the case of all orthogonally closed subspaces of S.

Let $\bar{S}$ denote the completion of S and let ϕ be a mapping from $E(S)$ into $E(\bar{S})$ defined via $\phi(M)=\bar{M}$, where $\bar{M}$ denotes the completion of M. Then (i) ϕ is injective; (ii) $\phi(M^{\perp})=\phi(M)^{\perp_{\bar{S}}}$, $M\in E(S)$, where $\perp_{\bar{S}}$ denotes the orthocomplementation in $\bar{S}$; (iii) $\phi(M)\perp\phi(N)$ whenever $M\perp N$, and $\phi(M\vee N)=\phi(M)\vee\phi(N)$.

Therefore, if m, is a charge (finitely additive measure) on $E(\bar{S})$, so is $m\circ\phi: M\mapsto m(\bar{M})$, $M\in E(S)$, on $E(S)$. However, we have (Dvurečenskij and Pulmannová, 1988, 1989) that for incomplete $S, m\circ\phi$ is never completely additive even if m is on $E(\bar{S})$. In particular, let T be a nonzero

Hermitian operator of trace class in $\bar{S}$; then the mapping (1) is always a completely additive charge on $E(\bar{S})$ and for incomplete S, the map

$$m(M) = \mathrm{tr}(TP^{\bar{M}}), \qquad M \in E(S) \tag{4}$$

is only a Jordan, $P(S)$-regular charge (Dvurečenskij, 1991, and to appear). Moreover, the following generalization (Dvurečenskij, 1991, and to appear) of the Aarnes (1970) theorem holds.

Theorem 2.1. Every Jordan charge m on $E(S)$, $\dim S \neq 2$, can be uniquely expressed as a sum of a Jordan $P(S)$-regular charge m_1 and a Jordan charge m_2 vanishing on $P(S)$. Any Jordan charge on $E(S)$ is $P(S)$-regular iff it is of the form (4) for some Hermitian trace operator T in $\bar{S}$.

We recall that if m is positive, so are m_1 and m_2.

Let $\Omega(S)$, $\Omega_r(S)$, and $\Omega_{ca}(S)$ denote the sets of all states, $P(S)$-regular states, and completely additive states, respectively, on $E(S)$. The set $\Omega(S)$ is always a nonempty convex set: For any unit vector $x \in S$, the mapping

$$m_x(M) = \|x_M\|^2, \qquad M \in E(S) \tag{5}$$

where $x = x_M + x_{M^\perp}$, $x_M \in M$, $x_{M^\perp} \in M^\perp$, is a $P(S)$-regular state on $E(S)$. In an analogous way, for any unit vector $x \in \bar{S}$ the mapping

$$m_x(M) = \|P^{\bar{M}} x\|^2, \qquad M \in E(S) \tag{6}$$

is a $P(S)$-regular state on $E(S)$.

By $J(S)$, $J_r(S)$, $J_\sigma(S)$, and $J_{ca}(S)$ we denote the sets of all Jordan charges, Jordan $P(S)$-regular charges, Jordan signed measures, and Jordan completely additive signed measures on $E(S)$, respectively. Analogously, we define $W(S)$, $W_r(S)$, $W_\sigma(S)$, and $W_{ca}(S)$ as the sets of all charges, $P(S)$-regular charges, signed measures, and completely additive signed measures, respectively, on $E(S)$. The following assertion holds.

Theorem 2.2. The following conditions hold:

1. S is complete iff $\Omega_{ca}(S) \neq \varnothing$.
2. S is complete iff $W_{ca}(S) \neq \{0\}$.
3. If $\dim S = \infty$, then $J_{ca}(S) = W_{ca}(S)$.
4. $J_r(\bar{S}) = J_{ca}(\bar{S}) = W_{ca}(\bar{S})$ if $\dim S = \infty$.

Proof. Condition 1 has been proved in Dvurečenskij and Pulmannová (1988) and Condition 2 in Dvurečenskij and Pulmannová (1989). Condition 3 follows from the results of Dorofeev and Sherstnev (1990), which have shown that any completely additive measure, when S is an infinitely dimensional Hilbert space, is bounded; and Condition 4 follows from Dvurečenskij (to appear). ■

We note that for any S, $\dim S \geq 3$, $J(S)$ is a proper subset of $W(S)$. As has been shown in Dvurečenskij (to appear), let ψ be any additive discontinuous functional from R into R, and let T be a nonconstant Hermitian operator of trace class in $\bar{S}$. Then the mapping $m: E(S) \to R$ defined via

$$m(M) = \psi(\operatorname{tr}(TP^{\bar{M}})), \qquad M \in E(S) \tag{7}$$

is a finitely additive charge which is not Jordan.

A nonempty subset $\mathscr{M}$ of $\Omega(S)$ is said to be a *strong system* of states if the proposition "if $m(M)=1$, then $m(N)=1$, $m \in \mathscr{M}$" implies $M \subseteq N$. If $\mathscr{M}$ is a strong system of states, then by Gudder (1966), (i) $\mathscr{M}$ is order determining, i.e., $M \subseteq N$ iff $m(M) \leq m(N)$ for all $m \in \mathscr{M}$; (ii) for any $M \neq 0$ there is a state $m \in \mathscr{M}$ such that $m(M)=1$.

A subset $\mathscr{M}$ of $\Omega(S)$ is *convex* if for all $m, n \in \mathscr{M}$ and for any t, $0 \leq t \leq 1$, $tm + (1-t)m \in \mathscr{M}$. A state m is a *pure state* of $\mathscr{M}$ if the property $m = tm_1 + (1-t)m_2$ for $0 < t < 1$ implies $m_1 = m_2 = m$. If $\mathscr{M} \subseteq \Omega(S)$, then $\operatorname{Con}(\mathscr{M})$ denotes the convex hull of $\mathscr{M}$.

For a unit vector $x \in S$ ($x \in \bar{S}$), let P_x denote the one-dimensional subspace of S ($\bar{S}$) generated by x, and by P^x we denote the orthoprojector from $\bar{S}$ onto P_x.

Lemma 2.3. Let $\operatorname{Ext}(S)$ denote the set of all pure states on $E(S)$ and let $\mathscr{V}(\mathscr{S})$ be the set of all states of the form (6) on $E(S)$. Then any pure state is either a $P(S)$-regular state or a state vanishing on $P(S)$, whenever $\dim S \neq 2$, and

$$\mathscr{V}(\mathscr{S}) \subseteq \operatorname{Ext}(\mathscr{S}) \tag{8}$$

Proof. Let m be a pure state on $E(S)$. By Theorem 2.1, $m = tm_1 + (1-t)m_2$, where $0 \leq t \leq 1$, m_1 is a $P(S)$-regular state, and m_2 is a state vanishing on $P(S)$. Therefore, $t \in \{0, 1\}$.

Now we prove (8). Let x be a unit vector in $\bar{S}$ and define m_x via (6). Suppose that $m_x = tm_1 + (1-t)m_2$, for some $0 < t < 1$. Due to Theorem 2.1, $m_1 = m_1^1 + m_1^2$ and $m_2 = m_2^1 + m_2^2$, where m_1^1, m_2^1 are $P(S)$-regular finitely additive measures on $E(S)$ and m_1^2, m_2^2 are finitely additive measures vanishing on $P(S)$.

Assume that m_1^1 and m_2^1 are determined by trace operators T_1 and T_2 in $\bar{S}$, so that there are systems of orthonormal vectors $\{e_i\}$ and $\{f_j\}$ in $\bar{S}$ and nonnegative numbers $\{\lambda_i\}$, $\{\mu_j\}$ such that $m_1^1(M) = \sum_i \lambda_i \| P^{\bar{M}} e_i \|^2$ and $m_2^1(M) = \sum_j \mu_j \| P^{\bar{M}} f_j \|^2$, $M \in E(S)$.

The density of S in $\bar{S}$ gives a sequence of unit vectors $\{x_n\}$ of S such that $\|x - x_n\| \to 0$. Hence,

$$\lim_n m_x(P_{x_n}) = \lim_n \operatorname{tr}\{[tT_1 + (1-t)T_2] P^{x_n}\} = \operatorname{tr}\{[tT_1 + (1-t)T_2] P^x\} = 1$$

which gives $\mathrm{tr}(T_1 P^x) = 1 = \mathrm{tr}(T_2 P^x)$. Therefore

$$\sum_i \lambda_i |(x, e_i)|^2 = 1 = \sum_j \mu_j |(x, f_j)|^2$$

This is possible iff there is a unique i and a unique j such that $\mu_i = 1 = \lambda_j$ and $P_{e_i} = P_x = P_{f_j}$, so that $m_x = m_1 = m_2$. ∎

Any pure state m_x on $E(S)$, where x is a unit vector in S, is said to be a *purely pure state* on $E(S)$, and by $\mathrm{Ext}_p(S)$ we denote the set of all purely pure states on $E(S)$.

Lemma 2.4. A nonempty subset $\mathcal{M}$ of $\Omega(S)$, $\dim S \neq 2$, is a strong system iff $\mathrm{Ext}_p(S) \subseteq \mathcal{M}$.

Proof. It is evident that $\mathrm{Ext}_p(S)$ is a strong system of states on $E(S)$. Therefore, $\mathcal{M}$ containing all purely pure states is a strong system, too.

Conversely, suppose that $\mathcal{M}$ is a strong system. Let x be any unit vector of S. Then there is a state $m \in \mathcal{M}$ such that $m(P_x) = 1$. Let $m = m_1 + m_2$ be a decomposition of m into a regular part m_1 and a part m_2 vanishing on $P(S)$. Then $m(P_x) = m_1(P_x) = 1$, so that $m_2 = 0$. Hence, there exists a sequence of orthonormal vectors $\{u_i\}$ of $\bar{S}$ and nonnegative numbers $\{\lambda_i\}$ such that $m(M) = \sum_i \lambda_i \| P^{\bar{M}} u_i \|^2$. Therefore, there is a unique u_i such that $P_{u_i} = P_x$, which means that $m = m_x$. ∎

From Lemma 2.3 we have now the following obvious completeness criterion.

Criterion 2.5. S is complete iff any $P(S)$-regular pure state on $E(S)$, $\dim S \neq 2$, is a purely pure state. Then they are completely additive.

3. STATES WITH SUPPORT

A splitting subspace M of S is said to be a *support* of a state m if $m(N) = 0$ iff $N \perp M$. If a support of m exists, it is unique. It is known (Dvurečenskij, 1990) that any $P(S)$-regular state on $E(S)$ of a Hilbert space S, $\dim S \neq 2$, is always completely additive, and, therefore, by Maeda (1980), has a support. For incomplete S, this assertion is invalid, as has been shown in Dvurečenskij (1991). In this section, we shall give completeness criteria using $P(S)$-regular states with supports.

We recall that by the dimension of a splitting subspace M we mean the cardinality of any maximal orthonormal system (MONS, for short) in M.

Lemma 3.1. If a state m on $E(S)$ possesses a support M, then $\dim M \leq \aleph_0$.

Proof. Let m be a state on $E(S)$ with a support. Express m in the form $m=m_1+m_2$, where m_1 is a $P(S)$-regular part, and m_2 is a part vanishing on $P(S)$. Since $m_2(P_x)=0$ for any unit vector $x\in S$, we conclude that $m_1\neq 0$. There is a sequence of orthonormal vectors $\{x_i\}$ in $\bar{S}$ and a sequence of positive numbers $\{\lambda_i\}$ such that $m_1=\sum_i \lambda_i m_{x_i}$.

Suppose that M is a support of m; then $m_1(M^\perp)=0$ and for any i, $x_i\in\bar{M}$. Let $\{y_j\,;j\in J\}$ be a MONS in M. Then for any i, $\sum_{j\in J}|(x_i,y_j)|^2<\infty$, so that there is an at most countable subset $J_0\subseteq J$ such that $(x_i,y_j)=0$ for all i and all $j\in J\backslash J_0$, which means that $J=J_0$. ■

Proposition 3.2. For any incomplete S, $\Omega(S)$ has a $P(S)$-regular pure state which has no support in $E(S)$.

Proof. The incompleteness of S implies that, due to (Dvurečenskij (1989*b*), Gross (1990), Gudder (1975), and Gudder and Holland (1975) there is a MONS $\{y_i\}$ in S which is not an orthonormal basis (ONB, for short) in S. Therefore, $\{y_i\}$ can be completed by elements of $\bar{S}\backslash S$ to be an ONB in $\bar{S}$. In other words, there always exist two orthonormal vectors x and y such that $x\in\bar{S}\backslash S$ and $y\in S$.

Now we claim that a $P(S)$-regular pure state m_x has no support in $E(S)$. Actually, if M were a support of m_x, then for any $z\in M$, $z\neq 0$, we would have $(z,x)\neq 0$, and for any $u\in M^\perp$, $u\perp x$. On the other hand, for y we have the decomposition $y=y_1+y_2$, where $y_1\in M$ and $y_2\in M^\perp$. Calculate $0=(x,y)=(x,y_1)+(x,y_2)=(x,y_1)\neq 0$, a contradiction; consequently, m_x has no support in $E(S)$. ■

Remark 3.3. If an incomplete S has the property that for any nonzero $x\in\bar{S}\backslash S$ there is a unit vector $y\in S$, $y\perp x$, then m_x has no support in $E(S)$.

Proposition 3.4. If $\dim S>\aleph_0$, then S is complete if any $P(S)$-regular state on $E(S)$ has a support in $E(S)$.

Proof. The necessity is evident. For sufficiency let us suppose that S is incomplete. Then for any unit vector x in $\bar{S}\backslash S$ and for any MONS $\{y_i\}$ in S we have $\sum_i|(y_i,x)|^2\leq 1$, which means that at most countably many vectors y_i's are not orthogonal to x. Therefore, there is a unit vector y in S, $y\perp x$. The rest now follows from Remark 3.3. ■

Theorem 3.5. If $\dim S\neq 2$, the following statements are equivalent:

1. S is complete.
2. Any $P(S)$-regular state on $E(S)$ has a support in $E(S)$.
3. For any sequence $\{x_i\}$ of orthonormal vectors in S and all positive numbers $\{\lambda_i\}$, $\sum_i\lambda_i=1$, the state $\sum_i\lambda_i m_{x_i}$ has a support in $E(S)$.

4. For any infinite sequence $\{x_i\}$ of orthonormal vectors in S, the state $\sum_i m_{x_i}/2^i$ has a support in $E(S)$.

Proof. It is clear that $1\Rightarrow 2\Rightarrow 3\Rightarrow 4$. Now we prove that Condition 4 gives 1.

First of all we prove that if $m=\sum_i \lambda_i m_{x_i}$, where $\{x_i\}$ is an orthonormal system in S and $\{\lambda_i\}$ are positive numbers such that $\sum_i \lambda_i=1$, then m has a support in $E(S)$ iff $\bigoplus_i P_{x_i}\in E(S)$. Moreover, in this case $\bigoplus_i P_{x_i}$ is a support of m.

Indeed, if m is a support of m, then $m_{x_i}(M^\perp)=0$ for all i, so that $x_i\perp M^\perp$, i.e., $x_i\in M$ for any i. If there is an $x\in M$, $x\perp x_i$ for all i, then $m(P_x)=0$, so that $P_x\perp M$, which means that $\{x_i\}$ is a MONS in M, and $M=\bigoplus_i P_{x_i}$.

On the other hand, if $M=\bigoplus_i P_{x_i}\in E(S)$, then $m(N)=0$ implies $m_{x_i}(N)=0$ or all i, and, therefore, $x_i\perp N$, so that $M\perp N$.

Using this property, we see that Condition 4 asserts that for any sequence of orthonormal vectors $\{x_i\}$, $\bigoplus_i P_{x_i}$ is a splitting subspace of S. In view of the criterion in Dvurečenskij (1988), this is equivalent to the completeness of S. ■

Example 3.6. Let H be a separable Hilbert space with an orthonormal basis $\{e_i\}$. Let $f=\sum_{i=1}^{\infty} e_i/2^i$ and let S be a linear subspace generated by $\{f, e_2, e_3, \ldots\}$. Then m_{e_1} is a $P(S)$-regular pure state on $E(S)$ having no support in $E(S)$.

4. EXPECTATION FUNCTIONALS

In this section, we give a completeness criterion using expectation functionals. For this and the following sections, we introduce the next notions from quantum logic theory.

By a *quantum logic* L we mean a poset L with a partial ordering $\leq$, maximal and minimal elements 0 and 1, respectively, and a unary operation $\perp: L\to L$ such that (i) $(a^\perp)^\perp=a$ for any $a\in L$; (ii) $a\vee a^\perp=1$ for any $a\in L$; (iii) if $a\leq b$, then $b^\perp\leq a^\perp$; (iv) if $a\leq b^\perp$, then $a\vee b\in L$; (v) if $a\leq b$, then $b=a\vee(b\wedge a^\perp)$ (orthomodular law). Two elements a, b of L are *orthogonal* and we write $a\perp b$ if $a\leq b^\perp$. A logic L which is closed with respect to the join of any sequence of mutually orthogonal elements is said to be a σ-logic.

For any inner product space S, $E(S)$ is a logic, and $E(S)$ is a σ-logic iff S is complete (Dvurečenskij, 1988).

A *state* is a mapping $m: L\to[0, 1]$ such that $m(1)=1$ and $m(a\vee b)=m(a)+m(b)$ for $a\perp b$. By $\Omega(L)$ we denote the set of all states on L; in general, it can be empty. As for $E(S)$, we define the notions as σ-additive state and completely additive state.

A mapping x from the Borel sets $B(R)$ into L such that (i) $x(\varnothing)=0$, (ii) if $E\cap F=\varnothing$, then $x(E)\perp x(F)$ and $x(E\cup F)=x(E)\vee x(F)$, and (iii) $x(R\backslash E)=x(E)^{\perp}$ for any $E\in B(R)$, is said to be an *observable*. If for an observable x we have that if $\{E_i\}$ is a sequence of mutually disjoint subsets of $B(R)$, then $x(\bigcup_i E_i)=\bigvee_i x(E_i)$, then x is said to be a *σ-observable*. An observable is said to be bounded if there is a compact set C such that $x(C)=1$. If x is an observable and m is a state on L, then $m\circ x$ is a probability state on $B(R)$, and $E(x;m)$ denotes the expectation value of an observable x in a state m defined via $E(x;m)=\int_R t\,dm\circ x(t)$. Some basic properties of $E(x;m)$ are investigated in Rüttimann (1985).

The following lines are very close to so-called Gleason-type problems in the context of quantum logics, as has been observed in Rüttimann (1989).

Let $\mathcal{M}$ be a nonempty convex poset of $\Omega(L)$ and define $J(M):=\operatorname{lin}(\mathcal{M})$. Then $\mathcal{M}$ is a base of a generating cone in $J(\mathcal{M})$ and the pair $(J(\mathcal{M}),\mathcal{M})$ is a base-norm space (Rüttimann, 1984). The corresponding base norm is denoted by $\|\cdot\|_{\mathcal{M}}$. We now follow the general theory of base-normed and order unit normed spaces (Alfsen, 1972): If we order the Banach dual $J^*(\mathcal{M})$ via $f\leq g$, $f, g\in J^*(\mathcal{M})$, iff $f(m)\leq g(m)$ for all $m\in\mathcal{M}$, then $(J^*(\mathcal{M}),\leq,1_{\mathcal{M}})$, where $1_{\mathcal{M}}$ is the unique linear functional with $1_{\mathcal{M}}:\mathcal{M}\to\{1\}$, is an order unit normed space, i.e., an Archimedean ordered vector space with the order unit $1_{\mathcal{M}}$. For the norm in $J^*(\mathcal{M})$, also denoted by $\|\cdot\|_{\mathcal{M}}$, we have

$$\|f\|_{\mathcal{M}}=\sup\{|f(m)|: m\in\mathcal{M}\}=\inf\{t>0: f\in t[-1_{\mathcal{M}},1_{\mathcal{M}}]\}$$

Note that $-\|f\|_{\mathcal{M}}\cdot 1_{\mathcal{M}}\leq f\leq\|f\|_{\mathcal{M}}\cdot 1_{\mathcal{M}}$, thus $[-1_{\mathcal{M}},1_{\mathcal{M}}]$ is the norm-closed unit ball in $J^*(\mathcal{M})$.

With any element $p\in L$ we associate a linear functional $e_{\mathcal{M}}(p)$ on $J(\mathcal{M})$ as follows: $e_{\mathcal{M}}(p)(m):=m(p)$, $m\in\mathcal{M}$. Since $e_{\mathcal{M}}(p)(\mathcal{M})\subseteq[0,1]$, we conclude that $e_{\mathcal{M}}(p)\in[0,1_{\mathcal{M}}]$.

An affine functional on $\mathcal{M}$ with the range in $[0,1]\subset R$ is called a *counter* on $\mathcal{M}$. This functional admits a unique extension to a linear functional on $J(\mathcal{M})$, is $\|\cdot\|_{\mathcal{M}}$-continuous, and belongs to the order interval $[0,1_{\mathcal{M}}]$. Conversely, the restriction of an element of $[0,1_{\mathcal{M}}]$ to $\mathcal{M}$ is a counter on $\mathcal{M}$.

For example, if $\mathcal{M}$ is a convex subset of $\Omega(L)$, and x is a bounded observable, then $E_{\mathcal{M}}:\mathcal{M}\to R$ defined via $E_{\mathcal{M}}(m)=E(x;m)$, $m\in\mathcal{M}$, can be uniquely extended to a $\|\cdot\|_{\Omega(L)}$-continuous linear functional on $J(\Omega(L))$.

We say that a counter on a convex subset $\mathcal{M}$ of $\Omega(L)$ is called *expectational*, respectively *σ-expectational*, if it is the restriction to $\mathcal{M}$ of the expectational functional of some bounded observable, respectively σ-observable.

Now we formulate the following completeness criterion.

Theorem 4.1. S is complete iff there is a strong system $\mathcal{M}$ of states on $E(S)$, $\dim S\neq 2$, such that any counter on $\operatorname{Con}(\mathcal{M})$ is σ-expectational. In this case $\mathcal{M}\subseteq\Omega_{ca}(S)$.

Proof. Let S be complete, and let $\mathscr{M}$ be the set of all completely additive states on $E(S)$. Then all elements of $J(\mathscr{M})$ are expressible by (4), where T is any Hermitian trace operator. It is well known that $J^*(\mathscr{M})$ is the set of all Hermitian operators on S because all observables on $E(S)$ (for a complete, S) are in a one-to-one correspondence with the set of all Hermitian operators in S.

According to Theorem 3.5 of Rüttimann (1985), we have $\mathscr{M}\subseteq\Omega_{ca}(S)$ and $E(S)$ is a complete lattice, which, in view of criteria in Dvurečenskij (1988, 1989*b*), means the completeness of S. ■

Remark 4.2. According to Lemma 2.4, $\mathscr{M}$ has to contain all purely pure states on $E(S)$.

5. QUADRATIC SPACES

The results of the previous section can be generalized to more general inner product spaces as real or complex ones.

Let K be a *-field with an involution $*: K\to K$ which satisfies $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$, $x^{**}=x$ for all $x, y\in K$. Let Φ be a bilinear form on a (left-) vector space S over a *-field K, i.e., a mapping $\Phi: S\times S\to K$ which satisfies

$$\Phi(\alpha x+\beta y, z)=\alpha\Phi(x, z)+\beta\Phi(y, z)$$

$$\Phi(x, \alpha y+\beta z)=\Phi(x, y)\alpha^*+\Phi(x, z)\beta^*$$

for all $x, y, z\in S$, $\alpha, \beta\in K$. The bilinear form Φ is Hermitian if $\Phi(x, y)=\Phi(y, x)^*$ for all $x, y\in K$ and Φ is anisotropic if $\Phi(x, x)=0$ implies $x=0$.

The couple (S, Φ), where Φ is a Hermitian anisotropic bilinear form, is said to be a *quadratic space*. Two vectors x and y of S are said to be Φ-orthogonal if $\Phi(x, y)=0$. For any $M\subseteq S$, $M^\perp:=\{x\in S: \Phi(x, y)=0$ for all $y\in M\}$. We define the set of all Φ-splitting subspaces of S, $E_\Phi(S)=\{M\subseteq S: M+M^\perp=S\}$, and the set of all Φ-orthogonally closed subspaces of S, $F_\Phi(S)=\{M\subseteq S: M^{\perp\perp}=M\}$. Then $E_\Phi(S)\subseteq F_\Phi(S)$, and $F_\Phi(S)$ is a complete, irreducible, orthocomplemented, atomic lattice with the covering property which is not orthomodular, in general. $E_\Phi(S)$ is always a quantum logic which is not a σ-logic, in general.

We recall that for any nonzero vector $x\in S$, by P_x we mean a one-dimensional subspace of S spanned over x.

We say that a quadratic space (S, Φ) is *orthomodular* iff $E_\Phi(S)=F_\Phi(S)$. In view of the criterion of Amemiya and Araki (1966–1967), a real or complex inner product space S is orthomodular iff S is complete.

Now we present an orthomodularity criterion generalizing that in Dvurečenskij (1988).

Theorem 5.1. A quadratic space (S, Φ) is orthomodular iff for any system of mutually Φ-orthogonal vectors $\{x_i\}$ of S, $\bigoplus_i P_{x_i} \in E_\Phi(S)$.

Proof. The necessity is evident. Suppose the sufficiency, i.e., $\{x_i\}^{\perp\perp}$ is an element of $E_\Phi(S)$ for any system of mutually Φ-orthogonal elements $\{x_i\}$ of S. Let M be a given element of F_Φ and choose a maximal set of nonzero orthogonal vectors in M, $\{x_i\}$. Then $M_0 := \{x_i\}^{\perp\perp} \subseteq M$. Let x be any arbitrary vector of M. Then $x = x_1 + x_2$, where $x_1 \in M_0$ and $x_2 \in M_0^\perp$. The maximality of $\{x_i\}$ gives $x_2 = 0$, and $x = x_1 \in M_0$, so that $M \in E_\Phi(S)$, and $F_\Phi(S) = E_\Phi(S)$. ■

The measure-theoretic criterion of the orthomodularity of a quadratic space (Dvurečenskij *et al.*, 1990) shows that under some conditions, (S, Φ) is orthomodular if $F_\Phi(S)$ possesses at least one state with a one-dimensional support.

Below we give a generalization of Theorem 4.1.

Theorem 5.2. A quadratic space (S, Φ) is orthomodular whenever there is a convex, strong system $\mathcal{M}$ of states on $E_\Phi(S)$ such that any counter on $\mathcal{M}$ is σ-expectational. In this case, any state of $\mathcal{M}$ is completely additive.

Proof. Following Theorem 3.5 of Rüttimann (1985), we conclude that any state of $\mathcal{M}$ is completely additive, and $E_\Phi(S)$ is a complete lattice, which in view of Theorem 5.1 means the orthomodularity of (S, Φ). ■

6. OPEN PROBLEMS

In this section, we present some unsolved problems of measure theory on $E(S)$, and we give partial solutions to them.

We say that a net $\{m_\alpha\}$ of charges on $E(S)$ converges weakly to a charge m on $E(S)$ if $\lim_\alpha m_\alpha(M) = m(M)$ for any $M \in E(S)$. Dvurečenskij (1978) proved the Nikodým theorem for σ-logics: if $\{m_n\}$ is a sequence of signed measures on L with a finite limit $m(a) = \lim_\alpha m_n(a)$ for all $a \in L$, then m is a signed measure on L, too. Jajte (1972) showed this result for a sequence of signed measures of the form (1) for a separable Hilbert space. Using the results of Dorofeev and Sherstnev (1990) and Dvurečenskij (1978), we can reformulate the result of Jajte as follows.

Theorem 6.1 (Nikodým theorem). The space $W_{ca}(H)$, $\dim H = \infty$, is weakly sequentially complete for any Hilbert space H.

Proof. Let $\{m_n\}$ be a fundamental weak sequence from $W_{ca}(H)$ with a limit m. By Dorofeev and Sherstnev (1990) we have that any m_n is of the form (1) for some Hermitian operator T_n of trace class in H. In view of

Dvurečenskij (1978) m is countably additive. It is clear that there is a Hermitian operator T in H such that $m_n(P_f)=(T_n f,f)\to(Tf,f)=m(P_f)$ for any unit vector f in H.

Let $\{f_i : i\in I\}$ be an arbitrary ONB in H. Then $H=\bigoplus_{i\in I}P_{f_i}$, and for any $n\geq 1$, there is an at least countable subset I_n of I such that $m_n(P_{f_i})=0$ for $i\in I\setminus I_n$. Put $I_0=\bigcup_{m=1}^{\infty}I_n$ and $H_0=\bigoplus_{i\in I_0}P_{f_i}$. Then $m_n(H)=m_n(H_0)$ for all n, so that $m(H)=m(H_0)$, and $m(P_{f_i})=0$ for all $i\in I\setminus I_0$. Hence,

$$
\begin{aligned}
m(H)&=m(H_0)+m(H_0^{\perp})=m(H_0)=\sum_{i\in I_0}m(P_{f_i})\\
&=\sum_{i\in I_0}m(P_{f_i})+\sum_{i\in I\setminus I_0}m(P_{f_i})=\sum_{i\in I}(Tf_i,f_i)=\operatorname{tr}T=\operatorname{tr}(TP^H)
\end{aligned}
$$

In an analogous way we have $m(M)=\operatorname{tr}(TP^M)$, $M\in E(H)$. ■

We note that the Nikodým theorem is true for any Hilbert space if we consider the space of all charges of the form (1); moreover, for $\{m_n\}$ with the limit m we have the uniform complete additivity with respect to n; see Dvurečenskij (1978).

Problem 6.1. Is the space $W_r(S)$ ($J_r(S)$, $\Omega_r(S)$) weakly sequentially complete for any incomplete S?

Let m be a charge on $E(S)$. We say that m (i) is *bounded* if

$$\sup\{|m(M)|: M\in E(S)\}<\infty$$

(ii) *$P(S)$-bounded* if

$$\sup\{|m(M)|: M\in P(S)\}<\infty$$

(iii) *$P_1(S)$-bounded* if

$$\sup\{|m(M)|: M\in P_1(S)\}<\infty$$

where $P_1(S)$ is the set of all one-dimensional subspaces of S.

The formula (7) gives an example of a $P_1(S)$-unbounded charge on $E(S)$. An interesting result of Dorofeev and Sherstnev (1990) says that for a Hilbert space H, any element of $W_{ca}(H)$ is bounded [and, consequently, of the form (1)] whenever $\dim H=\infty$.

Problem 6.2. Is any $P(S)$-regular charge on $E(S)$, $\dim S=\infty$, necessarily bounded?

The partial answers are presented below.

Theorem 6.2. Any $P_1(H)$-bounded, $P(H)$-regular charge on $E(H)$ is a Jordan completely additive signed measure on $E(H)$.

Proof. Our assumptions guarantee that on any finite-dimensional subspace M of H there is a bounded bilinear form t_M on $M \times M$ such that $t_M(x, x) = m(P_x)$ for any unit vector $x \in M$. Therefore, there is a bounded bilinear form t on $H \times H$ such that $t(x, x) = m(P_x)$ for any unit vector x in H. Hence, there is a Hermitian operator T in H such that $t(x, x) = (Tx, x)$, $x \in H$.

Let $T = T^+ - T^-$, where T^+, T^- are positive operators and let $H = H^+ \oplus H^-$, where H^+ and H^- are positive and negative with respect to T. Therefore, the restrictions $m^+ := m \mid E(H^+)$ and $m^- := -m \mid E(H^-)$ are positive $P(H^+)$- and $P(H^-)$-regular charges on $E(H^+)$ and $E(H^-)$, respectively.

The regularity of m^+ gives that for any $M \in E(S)$ there is a nondecreasing sequence of finite-dimensional subspaces $\{M_n\}$ of M such that $m^+(M) = \lim_n m^+(M_n) = \lim_n \operatorname{tr}(T^+ P^{M_n}) \geq 0$ for any $M \in E(H^+)$. Consequently, by Dvurečenskij (1990), m^+ is completely additive and $m^+(M) = \operatorname{tr}(T^+ P^M)$, so that T^+ is a trace operator in H^+, hence, in H, too. Because the same is true for m^- and T^-, we conclude that T is a trace operator in H.

For any orthoprojector P^M we have $|\operatorname{tr}(TP^M)| < \infty$, which means that

$$|m(M)| = \left|\lim_n m(M_n)\right| \leq \left|\lim_n [\operatorname{tr}(T^+ P^{M_n}) + \operatorname{tr}(T_- P^{M_n})]\right|$$
$$\leq \operatorname{tr} T^+ + \operatorname{tr} T^- = \operatorname{tr}|T|$$

where $\{M_n\}$ is a suitable sequence of finite-dimensional, nondecreasing subspaces of M. The last assertion means that m is necessarily bounded.

Moreover,

$$m(H) = m(H^+) + m(H^-) = m^+(H^+) - m^-(H^-)$$
$$= \operatorname{tr}(T^+) - \operatorname{tr}(T^-) = \operatorname{tr} T = \operatorname{tr}(TP^H)$$

If we repeat all our above considerations for any Hilbert space $M \in E(H)$, we find a Hermitian operator $T_M : M \to M$ of trace class in M such that $m(M) = \operatorname{tr}(T_M) = \operatorname{tr}(T_M P^M)$. Choose an ONB $\{e_i\}$ in M. Then $m(M) = \sum_i (T_M e_i, e_i) = \sum_i (Te_i, e_i) = \operatorname{tr}(TP^M)$.

In other words, m is completely additive. ■

Problem 6.3. Is Theorem 6.2 valid for any $E(S)$?

Theorem 6.3 (Aarnes theorem). Any signed measure m on $E(H)$, $\dim H = \infty$, can be uniquely expressed as a sum of a $P(H)$-regular, Jordan charge (hence, completely additive) and a signed measure vanishing on any separable subspace of a Hilbert space.

Proof. Using the result of Dorofeev and Sherstnev (1990), we can show that m is a $P_1(H)$-bounded charge. In an analogous manner as in the proof of Theorem 6.2, we have a Hermitian operator T in H such that $m(P_x) = (Tx, x)$ for each unit vector $x \in H$.

We claim that T is a trace operator. Express T in the form $T = T^+ - T^-$ and $H = H^+ \oplus H^-$, where $T^+ : H^+ \to H^+$ and $T^- : H^- \to H^-$. Let $\{f_i : i \in I\}$ be any ONB in H^+ and $\{e_j : j \in J\}$ be an ONB in H^-. For any at most countable index subset Γ of I we have, due to the σ-additivity

$$\sum_{i\in\Gamma} (T^+ f_i, f_i) = \sum_{i\in\Gamma} (Tf_i, f_i) = \sum_{i\in\Gamma} m(P_{f_i}) = m\left(\bigoplus_{i\in\Gamma} P_{f_i}\right)$$

so that $\sum_{i\in I} (T^+ f_i, f_i) < \infty$ and

$$\sum_{i\in I} (T^+ f_i, f_i) = \sum_{i\in I} (T^+ f_i, f_i) + \sum_{j\in J} (T^+ e_j, e_j) < \infty$$

Hence, T^+ is a trace operator in H; analogously, we proceed with T^-, which entails that T is a trace operator, too.

Define a $P(H)$-regular Jordan charge (= completely additive signed measure) m_1 via (1) and putting $m_2 = m - m_1$, we obtain the decomposition of m in question.

The uniqueness of the decomposition is now evident. ■

Remark 6.4. If the dimension of H is an infinite nonmeasurable cardinal, then m is necessarily completely additive on $E(H)$. Theorem 6.3 is invalid for charges; see formula (7). The author does not know whether any signed measure on $E(H)$, $\dim H = \infty$, is necessarily bounded.

Any $E(S)$ can be embedded in a natural way into $E(\bar{S})$, so that $E(S)$ can be considered as a sublogic of the complete logic $E(\bar{S})$. There appears a natural question of the extensibility of states on $E(S)$ to states on $E(\bar{S})$. Due to Theorem 2.1, this problem is reduced to the following.

Problem 6.4. Is it possible to extend any state on $E(S)$ vanishing on $P(S)$ to a state on $E(\bar{S})$?

It is clear that this extended state must vanish on $P(\bar{S})$. This problem is equivalent to the following one.

Proposition 6.5. Any state on $E(S)$, $\dim S \neq 2$, can be extended to a state on $E(\bar{S})$ iff $\Omega_r(S)$ is dense in a weak topology of states in $\Omega(S)$.

Proof. Suppose that $\Omega_r(S)$ is weakly dense in $\Omega(S)$. Then for any state m on $E(S)$ there is a net of $P(S)$-regular states $\{m_\alpha\}$ such that $m(M) = \lim_\alpha m_\alpha(M)$ for any $M \in E(S)$. Consequently, there is a net $\{T_\alpha\}$ of von Neumann operators in $\bar{S}$ such that $m_\alpha(M) = \mathrm{tr}(T_\alpha P^{\bar{M}})$, $M \in E(S)$ for any α.

Since the weak topology of states corresponds to the product topology in $[0, 1]^{E(S)}$, respectively in $[0, 1]^{E(\bar{S})}$ for the case of $E(\bar{S})$, of compact spaces $[0, 1]$, we conclude that there is a subnet $\{T_{\alpha'}\}$ of a net $\{T_\alpha\}$ and a state $m_0 \in \Omega(\bar{S})$ such that $m_0(M) = \lim_{\alpha'} \operatorname{tr}(T_{\alpha'} P^M)$, $M \in E(\bar{S})$. It is evident that $m_0 | E(S) = m$.

Conversely, let any $m \in \Omega(S)$ have an extension, m_0 say, to a state on $E(\bar{S})$. Since $\Omega_{ca}(\bar{S})$ is weakly dense in $\Omega(\bar{S})$, we find a net $\{T_\alpha\}$ of von Neumann operators in $\bar{S}$ such that $\operatorname{tr}(T_\alpha P^M) \to m_0(M)$ for any $M \in E(\bar{S})$. Hence, $\operatorname{tr}(T_\alpha P^{\bar{M}})$ for any $M \in E(S)$. ■

A charge m on $E(S)$ has a *Hahn decomposition* if there are two mutually orthogonal splitting subspaces S^+ and S^-, $S^+ \oplus S^- = S$, such that $m | E(S^+) \geq 0$ and $m | E(S^-) \leq 0$. For $J_{ca}(\bar{S})$ $[= W_{ca}(\bar{S})]$, $\dim S = \infty$, any of its elements has a Hahn decomposition.

Problem 6.5. Has any element of $J_r(S)$, $\dim S = \infty$, a Hahn decomposition? What is their connection to the completeness?

We say that a state m on $E(S)$ is a *Jauch–Piron state* if for any $M, N \in E(S)$ with $m(M) = 1 = m(N)$, there exists a $P \in E(S)$, $P \subseteq M$, $P \subseteq N$ such that $m(P) = 1$. Any state with a support is a Jauch–Piron one.

Problem 6.6. Is any $P(S)$-regular state a Jauch–Piron one? What is their connection to the completeness of S?

ACKNOWLEDGMENTS

The author is very indebted to Prof. G. T. Rüttimann for valuable discussions on the present material during the author's stay in Berne. The author is also grateful to a referee for his valuable comments.

This paper was prepared with the support of the Alexander von Humboldt Foundation, Bonn.

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